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Gravitational Radiation Damping of the Spinning Rod for Arbitrary Rotational Speed

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Abstract

Within linearised general relativity the energy loss of the spinning rod due to its gravitational radiation damping is calculated numerically for arbitrary rotational speeds. Comparison shows that the classical low-velocity approximation is good until rotational speeds of the ends of the rod of about $\frac{1}{3}$ of the velocity of light.

1. *Introduction*

Recently in this Journal (Frehland, 1971, in the following cited as I) we have investigated the energy-momentum-stress-tensor and the (linearised) gravitational field of a spinning rod (dumbbell) without restriction upon the rod's angular velocity except that given by the velocity of light.

In this paper† we shall use the results of I in order to determine the energy loss of the rod (dumbbell) due to its gravitational self-interaction (radiation damping).

The calculation of energy-radiation from mechanical mass systems by gravitational waves is a classical problem of general relativity, which was first examined by Einstein himself (1916, 1918) as an application of linearised theory. Hereafter in 1922, Eddington discussed the special case of a spinning rod. The basis of these works was Einstein's pseudotensorial energy-expression for the gravitational field, with the help of which the radiation flux through a closed surface, lying in the wave-zone, was determined. This procedure yields, in the special case of the spinning rod and for low velocities, a radiated power P

$$P = \frac{32}{5} G \,.\, \frac{\omega^6}{c^5} \theta^2 \tag{1.1}$$

(θ = moment of inertia).

For low velocities the results of the short-range field calculations in this paper yield agreement between the energy change and the radiated power

† Notations and conventions as in I.

P, given by the approximate value (1.1). For high velocities the deviation from (1.1) is slight in the case of the homogeneous rod and more significant for the dumbbell. The high velocity results are determined numerically.

2. *Gravitational Self-Interaction of Systems in Nearly Periodic Motion*

Before calculating directly the energy change of the rod we will derive the relevant relations between the energy change and the radiation flux. For Einstein's energy-momentum-pseudotensor $t_\mu{}^\nu$ of the gravitational field hold the divergence relations

$$\left.\begin{aligned} &[(T_\mu{}^\nu + t_\mu{}^\nu)\sqrt{(-g)}]_{|\nu} = 0 \\ &g = \det g_{\mu\nu} \end{aligned}\right\} \tag{2.1}$$

From these equations follow non-tensorial integral conservation laws. For closed and in lowest (zeroth)-order periodic systems (e.g. the spinning rod), we can simplify these conservation laws essentially by performing an additional time integration over one period $T = t_2 - t_1$ and by restricting to the linear approximation. In this approximation the metric tensor $g_{\mu\nu}$ and the pseudotensor $t_\mu{}^\nu$ are in De Donder coordinates periodical in T (cf. I (2.1.5)). Hence we get†

$$\int_{t_1}^{t_2} dt \int \overset{1}{T}_\mu{}^4{}_{|4}\, d\overset{0}{V} = -\int_{t_1}^{t_2} dt \int \overset{1}{t}_\mu{}^m\, d\overset{0}{f}_m \tag{2.2}$$

in which $d\overset{0}{f}_m$ is the two-dimensional surface element in the Minkowski space. For $\mu = 4$ the integral on the right-hand side is regarded as the energy radiated away by the system during one period and the left-hand integral as the change of the material energy ($\Delta E(T)$). In the sense of this interpretation, as a consequence of the periodic motion of matter, (2.2) means the following: There are no tails and the energy radiated away from the system is equal to that lost by the source.

In the following we shall discuss the left integral in (2.2) for $\mu = 4$. From the covariant conservation laws

$$T_\mu{}^\nu{}_{\|\nu} = 0 \tag{2.3}$$

it is easy to deduce for this integral under application of Gauss' theorem and integration over the whole matter system the following relation

$$\int T^{44}{}_{|4}\, d\overset{0}{V} = \tfrac{1}{2}\int \gamma_{\kappa\rho|4} \overset{0}{T}{}^{\kappa\rho}\, d\overset{0}{V} =: \dot{E} \tag{2.4}$$

(e.g. Einstein, 1918, Cooperstock, 1967).

It is easy to see, that for the spinning rod, $\dot{E}$ is constant in time in linear approximation. Hence $\dot{E}$ describes the energy loss $\Delta E/T$ averaged over one period T.

† $\overset{0}{A}, \overset{1}{A}, \ldots$ denotes the zeroth, first, ..., order of a quantity A.

3. *Gravitational Field of the Spinning Rod*

In I we have determined the energy-momentum-stress-tensor $\overset{0}{T}{}^{\mu\nu}$ of the spinning rod (length $2R$) rotating in the $x-y$-plane about the z-axis

$$\overset{0}{T}{}^{\mu\nu} = \begin{pmatrix} \left(\rho\omega^2 y^2 - \frac{x^2}{r^2}\sigma^{rr}\right), & -\left(\rho\omega^2 xy + \frac{xy}{r^2}\sigma^{rr}\right), & 0, & -\rho c\omega y \\ -\left(\rho\omega^2 xy + \frac{xy}{r^2}\sigma^{rr}\right), & \left(\rho\omega^2 x^2 - \frac{y^2}{r^2}\sigma^{rr}\right), & 0, & \rho c\omega x \\ 0 & 0 & 0 & 0 \\ -\rho c\omega y & \rho c\omega x & 0 & \rho c^2 \end{pmatrix} \tag{3.1}$$

wherein $\rho = \rho_0/(1-\beta^2)$ is the mass density with reference to the inertial system at rest and ρ_0 the proper density of matter (timelike eigenvalue of $\overset{0}{T}{}^{\mu\nu}$) while

$$\sigma^{rr} = \frac{1}{r}\int_r^R \rho\omega^2 r_1{}^2\, dr_1 \tag{3.1a}$$

describes the (radial) stresses in the rod.

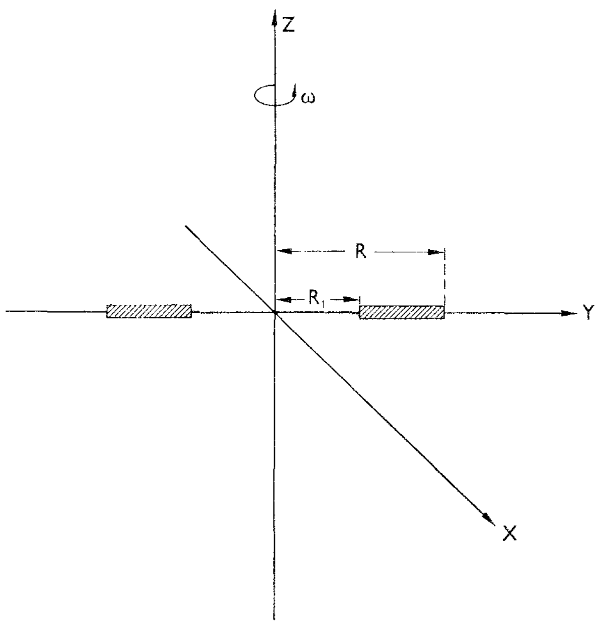

Figure 1.—Model of the spinning rod.

In order to avoid complication of the analysis we make the following simplifying assumptions for the mass distribution in the rod (see fig. 1):

$$\left.\begin{array}{lll} \text{(a)} & \rho = 0 & \text{for } r < R_1 \\ \text{(b)} & \rho = \text{const} & \text{for } R_1 \leqslant r \leqslant R \end{array}\right\} \tag{3.2}$$

(3.1) contains the limiting cases *homogenous rod* ($R_1 \to 0$) and *dumbbell* ($R_1 \to R$).†

Now we calculate the linearised field $\gamma_{\kappa\rho}$. Analogously to I, where we have determined the field only in the special case of the dumbbell, we evaluate the retarded integrals (I, (2.15)) as follows: We first determine the Liènard–Wichert potential of each single-mass element and then integrate over the rod. The result for the non-vanishing components $\gamma_{\kappa\rho}$ is:

$$\gamma_{xx} = -\frac{\kappa}{4\pi}\rho_L\left[\int_0^{R_1^*}\frac{dr}{\bar{R}'_{(r)}}\omega^2(R^2-R_1{}^2)(\tfrac{1}{2}-\sin^2\omega t'_{(r)}) + \int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}(\omega^2(r^2\cos^2\omega t'_{(r)} - R^2\sin^2\omega t'_{(r)} + R^2 - r^2) + c^2)\right]$$

$$\gamma_{yy} = -\frac{\kappa}{4\pi}\rho_L\left[\int_0^{R_1^*}\frac{dr}{\bar{R}'_{(r)}}\omega^2(R^2-R_1{}^2)(\tfrac{1}{2}-\cos^2\omega t'_{(r)}) + \int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}(\omega^2(r^2\sin^2\omega t'_{(r)} - R^2\cos^2\omega t'_{(r)} + R^2 - r^2) + c^2)\right]$$

$$\gamma_{zz} = -\frac{\kappa}{8\pi}\rho_L\left[\int_0^{R_1^*}\frac{dr}{\bar{R}'_{(r)}}\omega^2(R^2-R_1{}^2) + \int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}(2c^2+\omega^2(R^2-3r^2))\right]$$

$$\gamma_{xy} = +\frac{\kappa}{4\pi}\rho_L\left[\int_0^{R_1^*}\frac{dr}{\bar{R}'_{(r)}}\omega^2(R^2-R_1{}^2)\cos\omega t'_{(r)}\sin\omega t'_{(r)} + T\int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}\omega^2 R^2\cos\omega t'_{(r)}\sin\omega t'_{(r)}\right]$$

$$\gamma_{x4} = -\frac{\kappa}{2\pi}\rho_L c\int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}\omega r(\pm\sin\omega t'_{(r)})$$

$$\gamma_{y4} = +\frac{\kappa}{2\pi}\rho_L c\int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}\omega r(\pm\cos\omega t'_{(r)})$$

$$\gamma_{44} = -\frac{\kappa}{4\pi}\rho_L\left[\int_0^{R_1^*}\frac{dr}{\bar{R}'_{(r)}}\frac{\omega^2}{2}(R^2-R_1{}^2) + \int_{R_1}^{R_*}\frac{dr}{\bar{R}'_{(r)}}(c^2+\tfrac{3}{2}r^2-\tfrac{1}{2}R^2)\right] \tag{3.3}$$

† Of course this is only one possible definition for the *homogeneous rod.*

ρ_L = (constant) line density between R_1 and R, $t' = t - \bar{r}'/c$, $\bar{r}' = \bar{r}(t')$, $\bar{R}' = \bar{R}(t')$, $\bar{R} = \bar{r} \pm (\omega/c)r(x\sin\omega t - y\cos\omega t)$.

$\pm$ for (source) points on the right/left side of the rod ((r) denotes a point of the rod; $*$ means that integration over the left and right side of the rod is to be taken separately because of the retardation).

4. *Calculation of the Energy-Loss*

The computation of the constant value $\dot{E}$ according to (2.4) gets a considerable simplification, if we calculate it for the time $t = 0$, when the rod is lying in the x-axis. Then it follows from (2.4) and (3.1):

$$\dot{E} = \tfrac{1}{2}\int dV(\overset{0\;1}{\gamma}_{\kappa\rho|4}\overset{0}{T}{}^{\kappa\rho})_{(x,0,0,0)} = \tfrac{1}{2}\int dV\{-\overset{0\;1}{\gamma}_{xx|4}\,\sigma^{rr} + \overset{1}{\gamma}_{yy|4}\,\rho\omega^2 x^2 + 2\overset{1}{\gamma}_{y4|4}\,\rho c\omega x + \overset{1}{\gamma}_{44|4}\,\rho c^2\}_{(x,0,0,0)} \tag{4.1}$$

We evaluate (4.1) with the help of (3.1a) and (3.3).

(A) *Low Velocity Approximation*:

Firstly for low velocities we expand the integrand in (4.1) into powers of ω till the lowest non-vanishing order. A lengthy but elementary calculation yields:

$$\overset{1}{\gamma}_{xx|4}(x,0,0,0) = -\overset{1}{\gamma}_{yy|4}(x,0,0,0) = \frac{4}{3}\cdot\frac{\kappa\rho_L}{\pi c^2}\,\omega^4(R^3 - R_1{}^3) \tag{4.2a}$$

$$\overset{1}{\gamma}_{y4|4}(x,0,0,0) = \frac{8}{9}\frac{\kappa\rho_L}{\pi c^3}\,\omega^5(R^3 - R_1{}^3)\,x \tag{4.2b}$$

$$\overset{1}{\gamma}_{44|4}(x,0,0,0) = -\frac{8}{45}\frac{\kappa\rho_L}{\pi c^4}\,\omega^6(R^3 - R_1{}^3)\,x^2 \tag{4.2c}$$

If we substitute κ by Newton's gravitational constant G and use, that $\theta = \frac{2}{3}\rho_L(R^3 - R_1{}^3)$ is the moment of inertia of the rod, we get from (4.1) with (4.2) and (3.3):

$$\dot{E} = -\frac{32}{5}\frac{G\omega^6}{c^5}\,\theta^2 \tag{4.2}$$

This result is in fact in agreement with the radiated power P calculated by Einstein and Eddington (see (1.1)).

(B) *Arbitrary Rotational Speed*

We have calculated numerically the integral (4.1) for various $\omega R/c$ and R_1/R (see Fig. 1) and constant moment of inertia θ. The results are listed in Table 1 and drawn in Fig. 2.

Comparison shows that the low-velocity approximation is good until rotational velocities $\omega R/c$ of the ends of the rod of about $\frac{1}{3}$. For

TABLE 1

$\frac{\omega R}{c}$	$-\dot{E}\cdot\frac{1}{Gc\theta^2}$ Low velocity approximation	$\frac{R_1}{R}=0$	$\frac{R_1}{R}=\frac{15}{32}$	$\frac{R_1}{R}=\frac{23}{32}$	$\frac{R_1}{R}=\frac{31}{32}$
0.1	$0.16\cdot10^{-5}$	$0.16\cdot10^{-5}$	$0.16\cdot10^{-5}$	$0.16\cdot10^{-5}$	$0.16\cdot10^{-5}$
0.2	$0.10\cdot10^{-3}$	$0.10\cdot10^{-3}$	$0.10\cdot10^{-3}$	$0.10\cdot10^{-3}$	$0.10\cdot10^{-3}$
0.3	$0.12\cdot10^{-2}$	$0.11\cdot10^{-2}$	$0.11\cdot10^{-2}$	$0.11\cdot10^{-2}$	$0.11\cdot10^{-2}$
0.4	$0.07\cdot10^{-1}$	$0.06\cdot10^{-1}$	$0.06\cdot10^{-1}$	$0.06\cdot10^{-1}$	$0.06\cdot10^{-1}$
0.5	$0.25\cdot10^{-1}$	$0.23\cdot10^{-1}$	$0.23\cdot10^{-1}$	$0.23\cdot10^{-1}$	$0.23\cdot10^{-1}$
0.6	0.07	0.07	0.07	0.07	0.07
0.7	0.19	0.17	0.18	0.18	0.19
0.8	0.42	0.40	0.41	0.44	$0.05\cdot10$
0.9	$0.09\cdot10$	$0.09\cdot10$	$0.09\cdot10$	$0.10\cdot10$	$0.16\cdot10$
0.999	$0.16\cdot10$	$0.20\cdot10$	$0.22\cdot10$	$0.30\cdot10$	$0.22\cdot10^2$

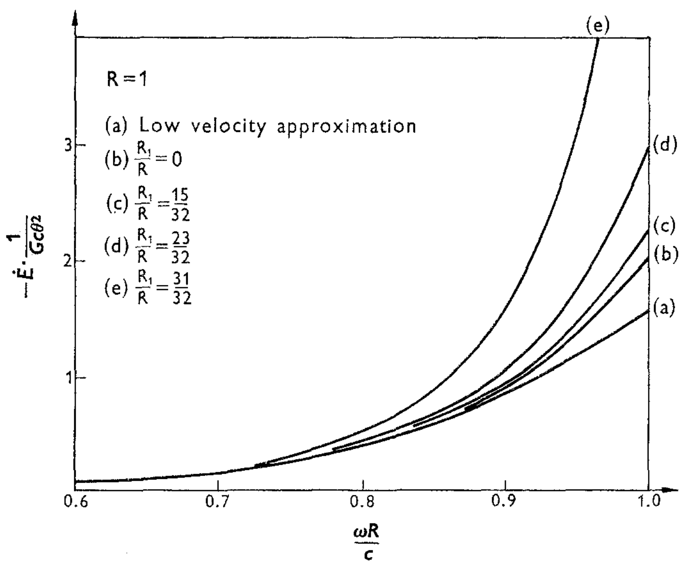

Table 1 and Figure 2.—The energy loss E as a function of the rotational velocity $\omega R/c$ of the ends of the rod for different values of the inner radius R_1 compared with the low velocity approximation (a).

$\omega R/c \to 1$ the deviations are slight in the case $R_1 = 0$ (rod) and become more significant for $R_1/R \to 1$ (dumbbell).

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References

Cooperstock, F. J. (1967). *Physical Review*, **163**, 1368.
Eddington, A. S. (1923). *Proceedings of the Royal Society*, **A102**, 268.
Einstein, A. (1916). *Sitzungsberichte der Preußischen*, Akademie der Wissenschaften, p. 688.
Einstein, A. (1918). *Sitzungsberichte der Preußischen*, Akademie der Wissenschaften, p. 155.
Frehland, E. (1971). *International Journal of Theoretical Physics*, **4**, 33.